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Note on the Theory of Optical Images.

By George Steić.

L. Mandelstam[*] obtains the result, that for a given aperture of a diaphragm only those linear structures will be similarly reproduced—through a lens-system—which satisfy the integral equation

$$f(\xi)=\lambda\int_{-a}^{+a} f(x)K(x-\xi)\,dx,$$

in the case of an illuminated structure; and the integral equation

$$f^2(\xi)=\lambda\int_{-a}^{+a} f^2(x)K^2(x-\xi)\,dx,$$

in the case of a self-luminous structure. Here $f(x)$ denotes the distribution of amplitudes in the linear object, or — briefly — the structure itself. $K(x-\xi)=K(s)$ is a function which depends only on the aperture of the diaphragm; when the aperture is circular, we have $K(s)=\frac{J_1(c\cdot s)}{s}$.[†] λ is a positive constant. $2a$ is the length of the linear object; the problem is more general, when the length of the structure is taken to be infinite.

Mandelstam assumes for the solution of the above equations

$$f(x)=\cos(bx+\phi),$$

which is permissible on certain suppositions. For the reproduced images we have the relation

$$\Phi(\xi)=\frac{1}{\lambda_b}f(x_b),$$

where $\frac{1}{\lambda_b}$ satisfies, for illuminated and self-luminous structures of infinite length, the equations

$$\frac{1}{\lambda_b}=2\int_0^\infty K(s)\cdot\cos bs\cdot ds$$

and

$$\frac{1}{\lambda_b}=2\int_0^\infty K^2(s)\cdot\cos bs\cdot ds$$

respectively.

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[*] Weber's *Festschrift* (1912), "Über eine Anwend. d. Integralgleich.," p. 237.

[†] $J_1(c\cdot s)$ = Bessel's function; $c=\frac{2\pi r}{l\cdot f}$ = const.; r = radius of the aperture; l = wave length of light; f = focus distance of the lens.

Mandelstam obtains explicit results for $\frac{1}{\lambda_b}$ in the case of a square aperture of the diaphragm, both for illuminated and for self-luminous structures; he obtains also an explicit result for a circular aperture, but only for illuminated structures. In the case of a circular aperture in the diaphragm and self-luminous structures of infinite length, he points out, with Lord Rayleigh,* that the solution of the problem depends on the evaluation of the integral

$$\int_0^\infty \left[\frac{J_1(c\cdot s)}{s}\right]^2 \cdot \cos bs \cdot ds, \tag{I}$$

which figures in the expression of $\frac{1}{\lambda_b}$ for self-luminous structures and circular aperture of the diaphragm. We easily recognize that such functions as

$$\left[\frac{J_1(c\,s)}{s}\right]^2 \cdot \cos b\,s \text{ or } \left[\frac{J_1(c\,s)}{s}\right]^2 \cdot \sin b\,s, \qquad J_1(c\,s) = \text{Bessel's function},$$

are finite and continuous for all values of s.

The purpose of this note is the evaluation of the former integral for the case $b \lesseqgtr 2c$.

We will start by taking into consideration the following integral:

$$\phi(\varepsilon) = c\int_0^\varepsilon \int_0^\infty \left[\frac{J_1(z)}{z}\right]^2 e^{i2\varepsilon z}\, dz\, d\varepsilon, \tag{II}$$

where $\varepsilon = \frac{b}{2c}$ and $i = \sqrt{-1}$, $J_1(z) =$ Bessel's function. For the present it is sufficient to suppose $0 \leq |\varepsilon| < M =$ finite constant. Considering fixed, for the moment, the upper limit of the outer integral, we can write

$$\phi(\varepsilon) = c\int_0^\infty \left[\frac{J_1(z)}{z}\right]^2 dz \int_0^\varepsilon e^{i2\varepsilon z}\, d\varepsilon. \tag{II'}$$

It is easy to see that

$$\text{(I)} = \text{real part of } \left[\frac{d\phi(\varepsilon)}{d\varepsilon}\right], \tag{III}$$

because from (II′) we get

$$\frac{d\phi(\varepsilon)}{d\varepsilon} = c\int_0^\infty \left[\frac{J_1(z)}{z}\right]^2 e^{i2\varepsilon z}\, dz,$$

the real part of which, after putting $z = cs$ and remembering $\varepsilon = \frac{b}{2c}$, gives (I).

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* *Sc. Pap.*, Vol. IV, p. 260.

We are going to transform the integral (II), and for this reason we take the well-known formula* existing for Bessel's functions,

$$J_n(as)\cdot J_n(bs)=\frac{s^n\cdot a^n\cdot b^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}\int_0^{\pi}\frac{J_n(s\sqrt{a^2+b^2-2ab\cos\phi})}{(\sqrt{a^2+b^2-2ab\cos\phi})^n}\sin^{2n}\phi\cdot d\phi.$$

Let us put here $n=1$, $a=b=c=\frac{2\pi r}{lf}$; further remembering that $\sqrt{1-\cos\phi}$ $=\sqrt{2}\sin\frac{\phi}{2}$, then putting $\frac{\phi}{2}=\psi$, and $\sin\psi=y$, we have

$$\left[\frac{J_1(cs)}{s}\right]^2=\frac{2c}{\sqrt{\pi}\cdot\Pi(\frac{1}{2})}\cdot\int_0^1\frac{J_1(s\cdot 2cy)}{s}y\cdot\sqrt{1-y^2}\,dy.$$

Multiplying this equation by e^{ibs} and integrating from 0 to ∞ with respect to s, we have

$$\int_0^{\infty}\left[\frac{J_1(cs)}{s}\right]^2e^{ibs}\,ds=\frac{2c}{\sqrt{\pi}\,\Pi(\frac{1}{2})}\int_0^1 y\cdot\sqrt{1-y^2}\,dy\int_0^{\infty}\frac{J_1(s\cdot 2cy)}{s}\cdot e^{ibs}\,ds.$$

Putting in the integral on the left $cs=z$ and remembering that $\frac{b}{c}=2\varepsilon$, then integrating on both sides from 0 to ε, we get

$$\begin{aligned}\phi(\varepsilon)&=c\int_0^{\varepsilon}\int_0^{\infty}\left[\frac{J_1(z)}{z}\right]^2e^{i2\varepsilon z}\,dz\,d\varepsilon\\&=\frac{2c}{\sqrt{\pi}\,\Pi(\frac{1}{2})}\int_0^{\varepsilon}d\varepsilon\int_0^1 y\sqrt{1-y^2}\,dy\int_0^{\infty}\frac{J_1(2cys)}{s}e^{ibs}\,ds.\qquad\text{(II}''\text{)}\end{aligned}$$

Let us consider

$$I_1=\int_0^{\infty}\frac{J_1(2cys)}{s}e^{ibs}\,ds.$$

Putting here $2cy\cdot s=t$ and considering y as constant during the integration, we get

$$I_1=\int_0^{\infty}\frac{J_1(t)}{t}e^{i\cdot\frac{\varepsilon}{y}\cdot t}\,dt.$$

But†

$$\int_0^{\infty}\frac{J_1(t)}{t}\cdot e^{ixt}\cdot dt=e^{i\arcsin x}\text{ for }0\leq x\leq 1.$$

So we will have

$$I_1=e^{i\arcsin\left(\frac{\varepsilon}{y}\right)},\qquad(1)$$

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* Nielsen, "Zylinder Funktionen, p. 183.

† Nielsen, *loc. cit.*, p. 197.

if $0 \leq \frac{\varepsilon}{y} \leq 1$.[*] In this inequality lies the definition of ε. Since y varies from 0 to 1, it is evident that the domain of variation for ε is

$$0 \leq \varepsilon = \frac{b}{2c} \leq 1.\dagger$$

So from (II″) we will have, after substituting (1),

$$\phi(\varepsilon) = \frac{2c}{\sqrt{\pi}\,\Pi(\frac{1}{2})} \int_0^{\epsilon} d\varepsilon \int_0^1 y\sqrt{1-y^2}\,dy \cdot e^{i \operatorname{arc\,sin}\left(\frac{\varepsilon}{y}\right)},$$

or

$$= \frac{2c}{\sqrt{\pi}\,\Pi(\frac{1}{2})} \int_0^{\epsilon} d\varepsilon \int_0^1 y\sqrt{1-y^2}\left[\cos\left(\operatorname{arc\,sin}\frac{\varepsilon}{y}\right) + i\sin\left(\operatorname{arc\,sin}\frac{\varepsilon}{y}\right)\right]dy. \quad \text{(II‴)}$$

Now we have ‡

$$\sin(\operatorname{arc\,sin} x) = (-1)^n \sin[(-1)^n\,|\operatorname{arc\,sin} x|]$$

and

$$\cos(\operatorname{arc\,sin} x) = (-1)^n \cos[(-1)^n\,|\operatorname{arc\,sin} x|].$$

The first of these two functions does not change its sign when n is odd or even; the second changes. But we have

$$\sin(|\operatorname{arc\,sin} x|) = x$$

and

$$\cos(|\operatorname{arc\,sin} x|) = \sqrt{1-x^2}.$$

Therefore

$$\sin(\operatorname{arc\,sin} x) = x$$

and

$$\cos(\operatorname{arc\,sin} x) = \pm\sqrt{1-x^2}.$$

So in our case

$$\sin\left(\operatorname{arc\,sin}\frac{\varepsilon}{y}\right) = \frac{\varepsilon}{y}$$

and

$$\cos\left(\operatorname{arc\,sin}\frac{\varepsilon}{y}\right) = \pm\sqrt{1-\frac{\varepsilon^2}{y^2}}.$$

And (II‴) becomes

$$\phi(\varepsilon) = \frac{2c}{\sqrt{\pi}\,\Pi(\frac{1}{2})} \int_0^{\epsilon} d\varepsilon \int_0^1 y\sqrt{1-y^2}\cdot\left[\pm\sqrt{1-\frac{\varepsilon^2}{y^2}} + i\frac{\varepsilon}{y}\right]dy. \quad \text{(II}^{\text{IV}}\text{)}$$

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[*] See Genocchi, "Diff.-Int. Rechnung," p. 170, for $\lim\limits_{\substack{\epsilon=0\\y=0}} \left(\frac{\epsilon}{y}\right) \geq 0$.

† Evidently, $M = 1$.

‡ Láska, "Formelsammlung," p. 2.

We will deal separately with the following two integrals:

$$\left.\begin{aligned} I_2 &= \int_0^1 y\sqrt{1-y^2}\sqrt{1-\frac{\varepsilon^2}{y^2}}\,dy, \\ I_3 &= \int_0^1 \sqrt{1-y^2}\,dy, \end{aligned}\right\} 0 \leq \varepsilon \leq 1.$$

After rearrangement we have

$$I_2 = i\,\varepsilon \int_0^1 \sqrt{1-y^2}\sqrt{1-\frac{y^2}{\varepsilon^2}}\,dy,$$

which is an elliptical integral. Further, we can write

$$I_2 = i\,\varepsilon\left[\int_0^1 \frac{dy}{\sqrt{W}} - \left(1+\frac{1}{\varepsilon^2}\right)\int_0^1 \frac{y^2\,dy}{\sqrt{W}} + \frac{1}{\varepsilon^2}\int_0^1 \frac{y^4\,dy}{\sqrt{W}}\right], \tag{2}$$

where $\sqrt{W} = \sqrt{(1-y^2)\left(1-\frac{y^2}{\varepsilon^2}\right)}$. Or briefly

$$I_2 = i\,\varepsilon\left[A - \left(1+\frac{1}{\varepsilon^2}\right)B + \frac{1}{\varepsilon^2}C\right],$$

the meaning of A, B and C being clear.

Now we can reduce the C by the well-known reduction formula *

$$(n-1)\,c\int\frac{y^n\,dy}{\sqrt{a+2by^2+cy^4}} = [y^{n-3}\sqrt{W}] - (n-3)\,a\int\frac{y^{n-4}}{\sqrt{W}}dy$$
$$-2(n-2)\,b\int\frac{y^{n-2}\,dy}{\sqrt{W}}.$$

Putting here $n=4$, $a=1$, $b=-\frac{1+\frac{1}{\varepsilon^2}}{2}$, $c=\frac{1}{\varepsilon^2}$ and taking into consideration the limits of our integral, we get

$$3\frac{1}{\varepsilon^2}C = -\int_0^1\frac{dy}{\sqrt{W}} + 2\left(1+\frac{1}{\varepsilon^2}\right)\int_0^1\frac{y^2\,dy}{\sqrt{W}} = -A + 2\left(1+\frac{1}{\varepsilon^2}\right)B,$$

and so (2) becomes

$$I_2 = i\,\varepsilon\left[\frac{2}{3}A - \frac{1}{3}\left(1+\frac{1}{\varepsilon^2}\right)B\right] \tag{2'}$$

But we can evaluate A and B. In

$$A = \int_0^1 \frac{dy}{\sqrt{1-y^2}\sqrt{1-\frac{y^2}{\varepsilon^2}}}$$

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* Legendre, "Traité des fonctions elliptiques," T. 1, p. 4.

put $y = \varepsilon x$ and we get

$$A = \varepsilon \int_0^{\frac{1}{\varepsilon}} \frac{dx}{\sqrt{1-\varepsilon^2 x^2}\,\sqrt{1-x^2}},$$

which is equal* to

$$A = \varepsilon\,(K + iK'),$$

where the K and K' are complete elliptic integrals of the first species, with the moduli ε and $\sqrt{1-\varepsilon^2}$ respectively. Further,

$$B = \int_0^1 \frac{y^2\,dy}{\sqrt{1-y^2}\sqrt{1-\frac{y^2}{\varepsilon^2}}}$$

becomes by the transformation $y = \varepsilon x$

$$B = \varepsilon \int_0^{\frac{1}{\varepsilon}} \frac{\varepsilon^2 x^2\,dx}{\sqrt{1-\varepsilon^2 x^2}\,\sqrt{1-x^2}} = \varepsilon \left\{ \int_0^1 + \int_1^{\frac{1}{\varepsilon}} \right\}.$$

But we can write†

$$B = \varepsilon\,[iE' + K - E],$$

where E and E' are complete elliptic integrals of the second species, with the moduli ε and $\sqrt{1-\varepsilon^2}$ respectively.

So (2) can be written

$$I_2 = i\varepsilon \left[\frac{2}{3}\varepsilon\,(K + iK') - \frac{1}{3}\left(1 + \frac{1}{\varepsilon^2}\right) \cdot \varepsilon\,(iE' + K - E) \right].$$

We get immediately

$$I_3 = \int_0^1 \sqrt{1-y^2}\,dy = \frac{\pi}{4}.$$

Substituting these values of I_2 and I_3 in ($\mathrm{II}^{\mathrm{IV}}$), we obtain

$$\phi(\varepsilon) = \frac{2c}{\sqrt{\pi}\,\Pi(\frac{1}{2})} \int_0^{\varepsilon} \left[\pm i\varepsilon \left\{ \frac{2}{3}\varepsilon\,(K + iK') - \frac{1}{3}\left(1 + \frac{1}{\varepsilon^2}\right) \cdot \varepsilon\,(iE' + K - E) \right\} + i\,\frac{\pi\varepsilon}{4} \right] d\varepsilon.$$

Separating the real and imaginary parts,

$$\phi(\varepsilon) = \frac{2c}{\sqrt{\pi}\,\Pi(\frac{1}{2})} \cdot \int_0^{\varepsilon} \pm \left[\frac{1}{3}(1+\varepsilon^2)\,E' - \frac{2\varepsilon^2}{3} K' \right] \cdot d\varepsilon + i \cdot \frac{2c}{\sqrt{\pi}\,\Pi(\frac{1}{2})} \int_0^{\varepsilon} \left[\frac{\pi\varepsilon}{4} \mp \left\{ \left(\frac{1-\varepsilon^2}{3}\right) K - \left(\frac{1+\varepsilon^2}{3}\right) E \right\} \right] d\varepsilon.$$

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* Durége, "Theorie d. ellipt. Funktionen," p. 91.

† Glaisher, *Quart. Journ.*, Vol. XX, p. 315.

Now remembering (II) and (III) and taking into consideration $\Pi(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$, we get

$$\left.\begin{aligned} I &= \int_0^\infty \left[\frac{J_1(cs)}{s}\right]^2 \cos bs\, ds = \frac{\pm 4c}{3\pi}\left[(1+\varepsilon^2)E' - 2\varepsilon^2 K'\right] \\ \text{and}\quad I' &= \int_0^\infty \left[\frac{J_1(cs)}{s}\right]^2 \sin bs\, ds = c\varepsilon \pm \frac{4c}{3\pi}\left[(1+\varepsilon^2)E - (1-\varepsilon^2)K\right], \end{aligned}\right\} \quad (\alpha)$$

where $0 \leq \left(\varepsilon = \frac{b}{2c}\right) \leq 1$.

In (α) we cannot retain the double sign, and we are going to decide which sign is to be used. As for the integration of $\phi(\varepsilon)$, we observe that we can use the following formulæ:

$$\left.\begin{aligned} (n+2)\int \varepsilon^n E\, d\varepsilon &= \varepsilon^{n+1}E + \int \varepsilon^n K\, d\varepsilon \\ \text{and}\quad (n+2)^2 \int \varepsilon^{n+2} K\, d\varepsilon &= \varepsilon^{n+1}E - (n+2)\varepsilon^{n+1}(1-\varepsilon^2)K + (n+1)^2 \int \varepsilon^n K\, d\varepsilon. \end{aligned}\right\} \quad (3)$$

If in (α) we put $\varepsilon = 0$, or, what is the same, $b = 0$, we get

$$\lim_{b=0} \int_0^\infty \left[\frac{J_1(cs)}{s}\right]^2 \cos bs\, ds = \int_0^\infty \left[\frac{J_1(cs)}{s}\right]^2 ds = \frac{\pm 4c}{3\pi},$$

since

$$\Pi(\tfrac{1}{2}) = \Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}, \quad \lim_{\epsilon=0} E' = 1, \quad \lim_{\epsilon=0} \varepsilon^2 K' = 0.$$

If we put $\nu = \mu = 1$ and $k = cs$, in the known formula*

$$\int_0^\infty \frac{J_\nu(k)\cdot J_\mu(k)}{k^{\nu+\mu}}\, dk = \frac{\Gamma(\frac{1}{2})\cdot\Gamma(\nu+\mu)}{2^{\nu+\mu}\cdot\Gamma(\nu+\frac{1}{2})\,\Gamma(\mu+\frac{1}{2})\cdot\Gamma(\nu+\mu+\frac{1}{2})},$$

we have

$$\int_0^\infty \left[\frac{J_1(cs)}{s}\right]^2 ds = \frac{+4c}{3\pi}.$$

The same result we obtain from† $Z = \int_v^\infty \frac{J_1^2(z)\, dz}{z\sqrt{z^2 - v^2}}$ by taking the limit for $v = 0$.

We see that $\lim_{\epsilon=0} I > 0$; on the other hand, we can show by simple expansion that for all values of ε the factor $[(1+\varepsilon^2)E' - 2\varepsilon^2 K']$ in the expression of I is always positive. Therefore, I is positive for all values of ε, and we must drop the negative sign from its expression. We will decide later on the choice of sign in I'.

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* Nielsen, *loc. cit.*, p. 194.

† Gray-Matthews, "Bessel's Functions," p. 201.

It remains to investigate the case for $\varepsilon = \frac{b}{2c} \geq 1$. To distinguish this case from the previous one, we will let $\varepsilon = \varepsilon'$, and we can write as before

$$\phi(\varepsilon') = c \int_1^{\varepsilon'} \int_0^{\infty} \left[\frac{J_1(z)}{z}\right]^2 e^{i2\varepsilon' z}\, dz\, d\varepsilon', \tag{IV}$$

where it is sufficient to suppose $|\varepsilon'| > 1$. Again,

$$(\mathrm{I}) = \text{real part of } \left[\frac{d\phi(\varepsilon')}{d\varepsilon'}\right]. \tag{V}$$

Omitting superfluous calculations, we can write immediately

$$\phi(\varepsilon') = \frac{4c}{\pi} \int_1^{\varepsilon'} d\varepsilon' \int_0^1 y\sqrt{1-y^2}\, dy \int_0^{\infty} \frac{J_1(2cy \cdot x)}{x} e^{ibx} \cdot dx. \tag{IV$'$}$$

Now

$$I_1 = \int_0^{\infty} \frac{J_1(2cy \cdot x)}{x} e^{ibx}\, dx = \int_0^{\infty} \frac{J_1(t)}{t} e^{\frac{\varepsilon'}{y}t}\, dt,$$

if $2cy \cdot x = t$ and $\frac{b}{2c} = \varepsilon'$. But*

$$\int_0^{\infty} \frac{J_1(t)}{t} e^{ixt} dt = i(x - \sqrt{x^2 - 1})$$

if $x \geq 1$. So we will have

$$I_1 = i\left[\frac{\varepsilon'}{y} - \sqrt{\left(\frac{\varepsilon'}{y}\right)^2 - 1}\right]$$

if $\frac{\varepsilon'}{y} \geq 1$. Since y varies between 0 and 1, in order that ε' should satisfy this inequality for all values of y, its domain of variation generally must be

$$\varepsilon' \geq 1.\dagger$$

So (IV′) can be written

$$\phi(\varepsilon') = i\frac{4c}{\pi} \int_1^{\varepsilon'} d\varepsilon' \int_0^1 y\sqrt{1-y^2}\left(\frac{\varepsilon'}{y} - \sqrt{\frac{\varepsilon'^2}{y^2} - 1}\right) dy. \tag{IV$''$}$$

We have to evaluate the integrals

$$I_2 = \int_0^1 \sqrt{1-y^2}\, dy = \frac{\pi}{4}$$

and

$$I_3 = \int_0^1 y \cdot \sqrt{1-y^2} \cdot \sqrt{\frac{\varepsilon'^2}{y^2} - 1} \cdot dy = \varepsilon' \int_0^1 \sqrt{1-y^2} \cdot \sqrt{1 - \frac{y^2}{\varepsilon'^2}} \cdot dy.$$

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* Nielsen, *loc. cit.*, p. 197.

† $\frac{\varepsilon'}{y} \geq 1$ could be satisfied, for $|y| < 1$, by $|\varepsilon'| < 1$; since our inequality must be satisfied for all values of y, even for $y = 1$, therefore we must take $|\varepsilon'| \geq 1$.

Introducing $\frac{1}{\varepsilon'^2} = \varepsilon''^2 \leq 1$ and applying the previous transformations and reductions, we get finally

$$I_3 = \varepsilon' \cdot \left[\frac{\varepsilon'^2-1}{3}K - \frac{\varepsilon'^2+1}{3}E\right],$$

where the modulus of K and E is $\frac{1}{\varepsilon'} = \varepsilon'' \leq 1$.

Substituting I_2 and I_3 in (IV″), we get

$$\phi(\varepsilon') = i\frac{4c}{\pi}\int_1^{\varepsilon'} \varepsilon' \left[\frac{\pi}{4} + \frac{\varepsilon'^2-1}{3}K - \frac{\varepsilon'^2+1}{3}E\right] d\varepsilon'.$$

We are able to evaluate $\phi(\varepsilon')$ with the aid of formulæ (3), but we are not interested in the explicit value of $\phi(\varepsilon')$; we are interested in $\frac{d\phi(\varepsilon')}{d\varepsilon'}$, in the equality

$$\frac{d\phi(\varepsilon')}{d\varepsilon'} = i\frac{4c}{\pi}\left[\varepsilon'\frac{\pi}{4} + \varepsilon'\frac{\varepsilon'^2-1}{3}K - \varepsilon'\frac{\varepsilon'^2+1}{3}E\right].$$

The real part of this expression is equal to zero; therefore, by (V),

$$\left.\begin{aligned} &Y = \int_0^\infty \left[\frac{J_1(cs)}{s}\right]^2 \cos bsds \equiv 0 \\ \text{and}\quad &Y' = \int_0^\infty \left[\frac{J_1(cs)}{s}\right]^2 \sin bsds = c\varepsilon' + \frac{4c\varepsilon'}{3\pi}\left[(\varepsilon'^2-1)K - (\varepsilon'^2+1)E\right], \end{aligned}\right\} \qquad (\beta)$$

where $\left(\varepsilon' = \frac{b}{2c}\right) \geq 1$ and the modulus in K and E is $\frac{1}{\varepsilon'}$.

If, in Y', we put $\varepsilon' = 1$, we get

$$\lim_{\varepsilon'=1} Y' = c - \frac{8c}{3\pi}.$$

On the other hand we have for I' in (α) for $\varepsilon = 1$

$$\lim_{\varepsilon=1} I' = c \pm \frac{8c}{3\pi}.$$

Since the integrand in I' is a continuous function, we must have

$$\lim_{\varepsilon'=1} Y' = \lim_{\varepsilon=1} I';$$

further, since, in I', $(1+\varepsilon^2)E - (1-\varepsilon^2)K > 0$ for all values of ε, we conclude that we must drop the positive sign in the expression of I'.

So we can finally write the formulæ (α),

$$\left.\begin{aligned} I &= \int_0^\infty \left[\frac{J_1(cs)}{s}\right]^2 \cos bs\, ds = \frac{4c}{3\pi}\left[(1+\varepsilon^2)E' - 2\varepsilon^2 K'\right], \\ I' &= \int_0^\infty \left[\frac{J_1(cs)}{s}\right]^2 \sin bs\, ds = c\varepsilon - \frac{4c}{3\pi}\left[(1+\varepsilon^2)E + (1-\varepsilon^2)K\right], \end{aligned}\right\} \qquad (\alpha')$$

where $0 \leqq \left(\varepsilon = \frac{b}{2c}\right) \leqq 1$.

It is interesting to investigate whether some of the functions I, I', Y, Y' show maxima or minima. We take I' and write, after putting $c = 1$, the condition for maxima and minima

$$\frac{dI'}{d\varepsilon} = 1 - \frac{4}{3\pi}\left[(1+\varepsilon^2)\frac{dE}{d\varepsilon} + 2\varepsilon E - (1-\varepsilon^2)\frac{dK}{d\varepsilon} + 2\varepsilon K\right] = 0.$$

Or, since*

$$\frac{dK}{d\varepsilon} = -\frac{1}{\varepsilon}\left(K - \frac{1}{1-\varepsilon^2}E\right),$$

and

$$\frac{dE}{d\varepsilon} = -\frac{1}{\varepsilon}(K - E),$$

we have

$$\frac{dI'}{d\varepsilon} = 1 - \frac{4}{\pi}\varepsilon E = 0.$$

Hence

$$\varepsilon E = \frac{\pi}{4} = 0.7854 = \frac{\pi}{2}\varepsilon\left\{1 - \left(\frac{1}{2}\right)^2\varepsilon^2 + 3\left(\frac{1}{2.4}\right)^2\varepsilon^4 - 5\left(\frac{1.3}{2.4.6}\right)^2\varepsilon^6 + \ldots\right\}.$$

By trial we find that $\varepsilon = 0.543$ approximately satisfies this equation. That this is a maximum point, we see immediately from

$$\frac{d^2I'}{d\varepsilon^2} = -\frac{4}{\pi}\left[2E(0.543) - K(0.543)\right].$$

From tables† we pick out the values of $E(0.543)$ and $K(0.543)$ and get

$$\frac{d^2I'}{d\varepsilon^2} = -\frac{4}{\pi}\left[2\times 1.445 - 1.712\right] < 0.$$

Similarly we find that neither I nor Y' has any maxima or minima.

What is the value of $\lim\limits_{\varepsilon'=\infty} Y'$? We put $c = 1$ and $\frac{1}{\varepsilon'} = \eta$ in Y', and get

$$\lim_{\eta=0} Y' = \lim_{\eta=0}\frac{1}{\eta^3}\left[\eta^2 - \frac{4}{3\pi}\left\{(1+\eta^2)E(\eta) - (1-\eta^2)K(\eta)\right\}\right] = \lim_{\eta=0}\frac{N}{D}.$$

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* Láska, Formelsammlung, p. 329.

† Jahnke, "Funkt. Tabl.," p. 68.

If we substitute $\eta = 0$, we get the indeterminate form $\frac{0}{0}$. By repeated application of the rules for the determination of such forms, we get finally

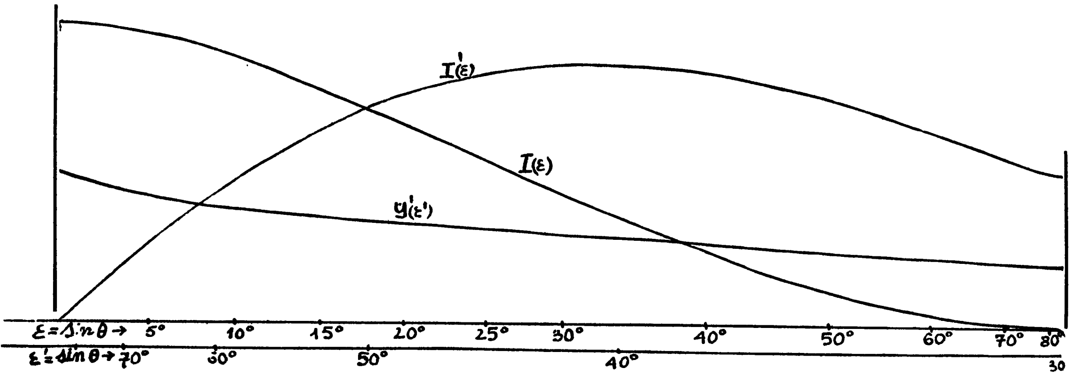
$$\lim_{\eta=0} Y' = \lim_{\varepsilon'=\infty} Y' = 0,$$

which shows that the curve representing the function Y' approaches asymptotically the ε'-axis.

The following tables contain numerical values for different values of ε and ε':

$\epsilon = \sin\theta$	$I(\epsilon)$	$I'(\epsilon)$	$\frac{1}{\epsilon'} = \sin\theta$	$Y'(\epsilon')$
$\theta = 0°$	$\frac{4c}{3\pi} \times 1$	0	$\theta = 0°$	0
5°	" $\times 0.960$	$0.079c$	5°	$0.011c$
10°	" $\times 0.882$	$0.140c$	10°	$0.012c$
15°	" $\times 0.776$	$0.192c$	15°	$0.023c$
20°	" $\times 0.667$	$0.228c$	20°	$0.038c$
25°	" $\times 0.545$	$0.249c$	25°	$0.047c$
30°	" $\times 0.435$	$0.258c$	30°	$0.060c$
40°	" $\times 0.247$	$0.252c$	40°	$0.089c$
50°	" $\times 0.112$	$0.225c$	50°	$0.106c$
60°	" $\times 0.040$	$0.194c$	60°	$0.115c$
70°	" $\times 0.007$	$0.169c$	70°	$0.130c$
80°	" $\times 0.001$	$0.154c$	80°	$0.146c$
90°	0	$0.150c$	90°	$0.150c$

Graphically:



In the figure we take, for unity of the ordinates, $\frac{4c}{3\pi} = 60$ mm. in the case of $I(\varepsilon)$; 200 mm. in the case of $I'(\varepsilon)$ and $Y'(\varepsilon')$. The line of abscissas containing $\varepsilon = \sin\theta$ refers to $I(\varepsilon)$ and $I'(\varepsilon)$; the one containing $\varepsilon' = \sin\theta$ refers to $Y'(\varepsilon')$.

We now come to the end of our investigation. In our problem we are concerned only with the integral

$$\frac{1}{\lambda_b}=2\int_0^\infty\left[\frac{J_1(cs)}{s}\right]^2\cos bs\,ds$$

for

$$0\leq\left(\varepsilon=\frac{b}{2c}\right)\leq 1,$$

or explicitly

$$\frac{1}{\lambda_b}=\frac{8c}{3\pi}\left[(1+\varepsilon^2)E'-2\varepsilon^2K'\right].$$

To every value of b will correspond a value of $\frac{1}{\lambda_b}$. We get the greatest value for $\frac{1}{\lambda_b}$ when $b=0$: $\frac{1}{\lambda_0}=\frac{8c}{3\pi}$. The smallest value of $\frac{1}{\lambda_b}$ is for $b=2c$: $\frac{1}{\lambda_{2c}}=0$. In this case we have no image. For details see Mandelstam's work.